On on-board indirect method for hybrid electric vehicle torque split and gear shift problem.

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Introduction

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- Olivier Flebus, Vitesco Technologies, Toulouse,
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- Serge Laporte, IMT, Toulouse,
- Mariano Sans, Vitesco Technologies, Toulouse.

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Cycle

We consider a Hybrid Electric Vehicle (HEV) on a predefined cycle, i.e. speed and slope trajectories are prescribed.

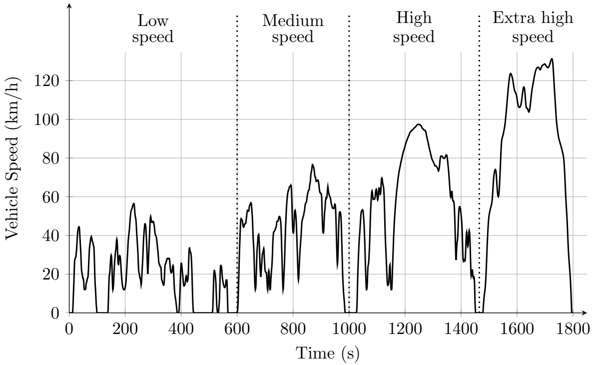


Figure: Worldwide harmonized Light vehicles Test Cycle (WLTC).

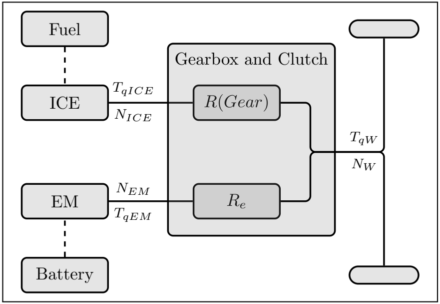
Requested wheels torque $T_{qW}(t)$ and rotation speed $N_W(t)$ are obtained with the information of our vehicle (mass, wheel diameter, aerodynamic coefficient. . .).

Static model of HEV

Inputs of our static model:

Name	Description	Unit
Cost		
m_{Fuel}	Fuel consumption	g
State		
SOC	Battery state of charge	
Commands		
$Gear$	Gearbox selector	
T_{qICE}	ICE torque	N.m
External inputs		
T_{qW}	Wheels torque	N.m
N_W	Wheels rotation speed	RPM

Figure: Schema of the HEV.



Outputs: $\dot{m}_{Fuel}$ and $\dot{SOC}$, where $\dot{}$ stands for $\dfrac{\mathrm{d}}{\mathrm{d}t}$.

HEV torque split and gear shift problem

The HEV torque split and gear shift problem can be formulated as a classical Lagrange optimal control problem

$$
\text{(OCP)} \quad \begin{cases} \displaystyle\min_{x,u} \int_{t_0}^{t_f} f^0\big(t, x(t), u(t)\big)\, \mathrm{d}t, & \\ \text{s.t. } \dot{x}(t) = f\big(t, x(t), u(t)\big), & t \in [t_0, t_f] \text{ a.e.}, \\ \qquad u(t) \in \mathrm{U}(t), & t \in [t_0, t_f], \\ \qquad x(t_0) = x_0, \quad x(t_f) = x_T, & \end{cases}
$$

where:

- $x \in \mathrm{AC}([t_0, t_f], \mathbb{R})$ corresponds to the *SOC*,
- $u \in \mathrm{L}^\infty([t_0, t_f], \mathbb{R}^2)$ corresponds to the pair $\big(T_{qICE}, Gear\big)$,
- functions f^0 and f are $\mathcal{C}^1$ w.r.t. x and u,
- $\mathrm{U}(t) \subset \mathbb{R}^2$ is a nonempty closed set for every $t \in [t_0, t_f]$, with regularity assumptions.[1]

[1](cf. [Cesari, 1983, Chapter 4.2, Remark 5] for more information)

Pontryagin's Maximum Principle

If (x, u) is solution of (OCP), there exists a costate $p \in \mathrm{AC}([t_0, t_f], \mathbb{R})$ and $p^0 \leq 0$ such that $(p^0, p) \neq 0$, the Hamilton's dynamic is satisfied for almost every $t \in [t_0, t_f]$:

$$\begin{cases} \dot{x}(t) = \nabla_p h\big(t, x(t), p(t), p^0, u(t)\big), \\ \dot{p}(t) = -\nabla_x h\big(t, x(t), p(t), p^0, u(t)\big), \end{cases} \tag{1}$$

and the maximization condition is satisfied for almost every $t \in [t_0, t_f]$:

$$h\left(t, x(t), p(t), p^0, u(t)\right) = \max_{w \in \mathrm{U}(t)} h\left(t, x(t), p(t), p^0, w\right), \tag{2}$$

where h is the pseudo-Hamiltonian defined by

$$h(t, x, p, p^0, u) = p^0\ f^0(t, x, u) + p\ f(t, x, u).$$

Definitions - Extremals

An extremal is a quadruplet

$$(x, p, p^0, u) \in \mathrm{AC}([t_0, t_f], \mathbb{R}) \times \mathrm{AC}([t_0, t_f], \mathbb{R}) \times \mathbb{R} \times \mathrm{L}^\infty([t_0, t_f], \mathbb{R}^2)$$

which satisfies the Hamilton's dynamic (1) and the maximization condition (2).

A BC-extremal is an extremal which satisfies the boundary conditions given by $x(t_0) = x_0$ and $x(t_f) = x_T$.

An extremal is said normal$^-$ if $p^0 < 0$, normal$^+$ if $p^0 > 0$ and abnormal if $p^0 = 0$.

Framework

Let us denote $\exp_{\vec{h}}(t_2, t_1, z_1, p^0)$ a solution at time t_2 of

$$\begin{cases} \dot{z}(t) = \vec{h}\left(t, z(t), p^0, u(t)\right), & t \in [t_1, t_2] \text{ a.e.} \\ h\left(t, z(t), p^0, u(t)\right) = \max\limits_{w \in \mathrm{U}(t)} h\left(t, z(t), p^0, w\right), & t \in [t_1, t_2] \text{ a.e.} \\ z(t_1) = z_1, & \end{cases}$$

where the pseudo-Hamiltonian vector field $\vec{h}$ is defined by

$$\vec{h}(t, x, p, p^0, u) = \left(\nabla_x h(t, x, p, p^0, u), -\nabla_p h(t, x, p, p^0, u)\right).$$

We consider the following hypothesis

Hypothesis 1

The possibly multivalued function $\exp_{\vec{h}}(t_2, t_1, x, p, p^0)$ is an application, defined for all $t_0 \leq t_1 < t_2 \leq t_f$, for all $x \in \mathbb{R}$ and for all non trivial $(p^0, p) \in \mathbb{R}^2$.

Simple shooting method

Under the previous hypothesis, the maximum principle leads to the resolution of

$$\text{(TPBVP)} \qquad \begin{cases} \pi_x\big(\exp_{\vec{h}}(t_f, t_0, z_0, p^0)\big) = x_T, \\ \pi_x(z_0) = x_0, \quad p^0 \le 0, \end{cases}$$

where $\pi_x(\cdot)$ is the classical x-space projection.

The simple shooting method aims to find a non-trivial zero (p^0, p) of the shooting function

$$\begin{array}{rccc} S : & \mathbb{R}^- \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ & (p^0, p) & \longmapsto & \pi_x\big(\exp_{\vec{h}}(t_f, t_0, x_0, p, p^0)\big) - x_T \end{array}$$

Normalization of the shooting function

Let us remark that if $(p^0, p) \neq 0$ satisfies $S(p^0, p) = 0$ then for all $k > 0$, $S(kp^0, kp) = 0$ (due to homogeneity of BC-extremals on (p^0, p)).

We propose two normalizations of the shooting function S.

- Method 1: if we assume that the extremals associated to a solution are normal$^-$ ($p^0 < 0$), then we can fix $p^0 = -1$ and consider $S_1 \colon \mathbb{R} \to \mathbb{R}$ defined by

$$S_1(p) = S(-1, p),$$

- Method 2: without the above assumption, we can fix $\|(p^0, p)\|_2 = 1$ and consider $S_2 \colon [-1, 1] \to \mathbb{R}$ defined by

$$S_2(p) = S\big(\eta(p), p\big), \quad \text{where} \quad \eta(p) = -\sqrt{1 - p^2}.$$

Goals

The application is an industrial problem and the method needs to be:

- fast,
- robust,
- computationally efficient.

Figure: Master controller.

Multiple shooting method

The time interval $[t_0, t_f]$ is decomposed into $\Delta_i = [t_i, t_{i+1}], i \in \mathbb{N}_N$, where $t_0 < t_1 < \cdots < t_N < t_{N+1} = t_f$ and $\mathbb{N}_N = \{0, \ldots, N\}$.

Problem (TPBVP) is transformed into

$$\text{(MPBVP)} \qquad \begin{cases} \forall i \in \mathbb{N}_{N-1}, \quad z_{i+1} = \exp_{\vec{h}}(t_i, t_{i+1}, z_i, p^0), \quad p^0 \leq 0, \\ \pi_x(z_0) = x_0, \quad \pi_x\big(\exp_{\vec{h}}(t_{N+1}, t_N, z_N, p^0\big) = x_T. \end{cases}$$

The multiple shooting function is defined by

$$\big(p_0, z_1, \ldots, z_N, p^0\big) \longmapsto \begin{pmatrix} \exp_{\vec{h}}\left(t_1, t_0, x_0, p_0, p^0\right) - z_1 \\ \exp_{\vec{h}}\left(t_2, t_1, z_1, p^0\right) - z_2 \\ \vdots \\ \exp_{\vec{h}}\left(t_N, t_{N-1}, z_{N-1}, p^0\right) - z_N \\ \pi_x\big(\exp_{\vec{h}}\left(t_{N+1}, t_N, z_N, p^0\right)\big) - x_T \end{pmatrix}.$$

This function is known to be less sensitive to the initial guess than the function S [Bock and Plitt, 1984].

Goals

Simple and multiple shooting are both optimal methods.

Nevertheless, compared to simple shooting, multiple shooting is

- faster,
- more robust,
- computationally equivalent.

Goal : propose a method which also reduces the number of computation.

Main idea[2]: the Macro-Micro method based on a bilevel decomposition of Problem (OCP).

[2] cf. [Cots et al., 2023a] for more information

Bilevel decomposition

Defining for all $i \in \mathbb{N}_N$ the intermediate optimal control problems

$$
(\mathrm{OCP}_{i,a,b}) \quad \begin{cases} V_i(a,b) := \min\limits_{x,u} \displaystyle\int_{t_i}^{t_{i+1}} f^0\big(t, x(t), u(t)\big)\,\mathrm{d}t, & \\ \text{s.t. } \dot{x}(t) = f\big(t, x(t), u(t)\big), & t \in \Delta_i \text{ a.e.}, \\ \qquad u(t) \in \mathrm{U}(t), & t \in \Delta_i, \\ \qquad x(t_i) = a, \quad x(t_{i+1}) = b, & \end{cases}
$$

where V_i corresponds to the value function, Problem (OCP) can be formulated into the equivalent form

$$
(\mathrm{BOCP}) \qquad \begin{cases} \min\limits_{X} V(X) := \displaystyle\sum_{i=0}^{N} V_i(X_i, X_{i+1}), \\ \text{s.t.} \quad X \in \mathcal{X}, \quad X_0 = x_0, \quad X_{N+1} = x_T, \end{cases}
$$

where $\mathcal{X}$ is the set of admissible intermediate states $X = \big(X_0, \ldots, X_{N+1}\big)$.

Commutative diagram

Under Hypothesis 1 and if

- the BC-extremals associated to (OCP) are normal$^-$ ($p^0 < 0$),
- the function V is differentiable at a solution of (BOCP),

then the following diagram is commutative

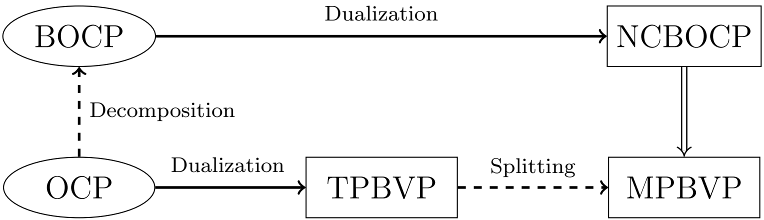


Figure: Diagram from (OCP) to (MPBVP).

Idea of proof

To prove this commutation, we mainly need the following result:

Under the previous assumption, if (x_i, u_i) is a solution of $(\mathrm{OCP}_{i,a,b})$, with $(x_i, p_i, -1, u_i)$ an associated BC-extremal, then we have

$$\nabla_a V_i\big(x_i(t_i), x_i(t_{i+1})\big) = -p_i(t_i), \qquad (3)$$

$$\nabla_b V_i\big(x_i(t_i), x_i(t_{i+1})\big) = p_i(t_{i+1}).$$

Main idea of the Macro-Micro method

Let us assume that the value functions V_i are known a priori.
We have to solve

- first the optimization problem

$$\begin{cases} \min\limits_{X} V(X) := \sum_{i=0}^{N} V_i\left(X_i, X_{i+1}\right), \\ \text{s.t. } X \in \mathcal{X}, \quad X_0 = x_0, \quad X_{N+1} = x_T, \end{cases}$$

  to get the optimal intermediate states $X^* = \left(X_0^*, \ldots, X_{N+1}^*\right)$,

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  to get the optimal intermediate states $X^* = \left(X_0^*, \ldots, X_{N+1}^*\right)$,

- and then the $N+1$ independent optimal control problems

$$\begin{cases} \min_{x,u} \int_{t_i}^{t_{i+1}} f^0\left(t, x(t), u(t)\right) \mathrm{d}t, \\ \text{s.t. } \dot{x}(t) = f\left(t, x(t), u(t)\right), & t \in \Delta_i \text{ a.e.}, \\ \qquad u(t) \in \mathrm{U}(t), & t \in \Delta_i, \\ \qquad x(t_i) = X_i^*, \quad x(t_{i+1}) = X_{i+1}^*, \end{cases}$$

  where $p^* = -\nabla_a V_i(X_i^*, X_{i+1}^*)$ is a zero of S_1, thanks to (3).

Proposed approach

The proposed approach is based on an approximation C_i of the value function V_i. We have to solve

- first the optimization problem

$$\text{(Macro)} \qquad \begin{cases} \min_{X} C(X) := \sum_{i=0}^{N} C_i\left(X_i, X_{i+1}\right), \\ \text{s.t. } X \in \mathcal{X}, \quad X_0 = x_0, \quad X_{N+1} = x_T, \end{cases}$$

  to get the "optimal" intermediate states $\hat{X} = \left(\hat{X}_0, \ldots, \hat{X}_{N+1}\right)$,

Proposed approach

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$$\text{(Macro)} \qquad \begin{cases} \displaystyle\min_{X} C(X) := \sum_{i=0}^{N} C_i\left(X_i, X_{i+1}\right), \\ \text{s.t. } X \in \mathcal{X}, \quad X_0 = x_0, \quad X_{N+1} = x_T, \end{cases}$$

  to get the "optimal" intermediate states $\hat{X} = (\hat{X}_0, \dots, \hat{X}_{N+1})$,

- and the $N+1$ independent optimal control problems

$$\text{(Micro)} \qquad \begin{cases} \displaystyle\min_{x,u} \int_{t_i}^{t_{i+1}} f^0\left(t, x(t), u(t)\right) \mathrm{d}t, \\ \text{s.t. } \dot{x}(t) = f\left(t, x(t), u(t)\right), & t \in \Delta_i \text{ a.e.}, \\ \qquad u(t) \in U(t), & t \in \Delta_i, \\ \qquad x(t_i) = \hat{X}_i, \quad x(t_{i+1}) = \hat{X}_{i+1}, \end{cases}$$

  where $p^* = -\nabla_a C_i(\hat{X}_i, \hat{X}_{i+1})$ is not necessary a zero of S_1.

Schema of the Macro-Micro method

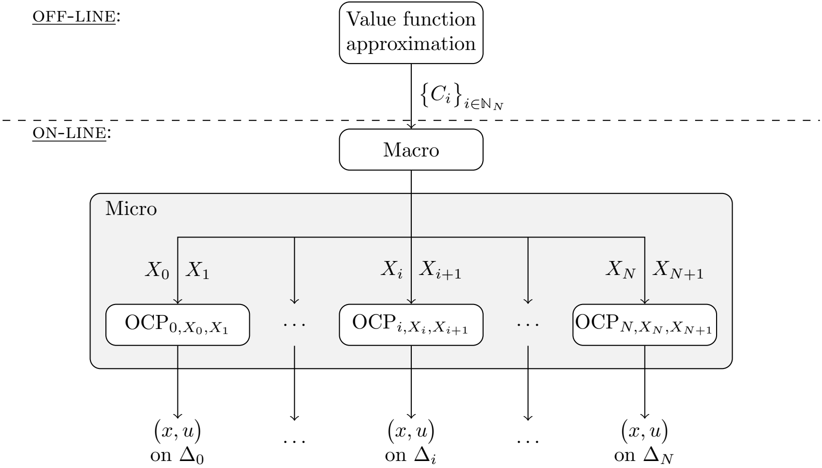

Figure: Schema of the Macro-Micro method.

Construction of the approximation of the value function

For all $i \in \mathbb{N}_N$, a database $\mathbb{D}_i$ of value function evaluations is constructed by an efficient method[3] only based on the computation of $\exp_{\vec{h}}$ instead of the evaluation of V_i.

The functions C_i are modeled by neural networks.

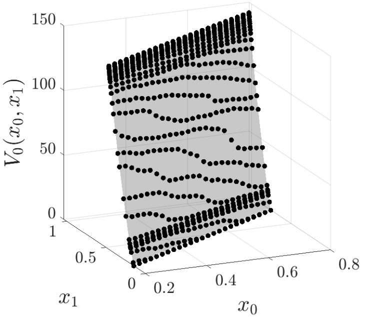

Figure: The points correspond to $\mathbb{D}_0$ and the surface to the neural network C_0.

[3] cf. [Cots et al., 2023b] for more information

Results

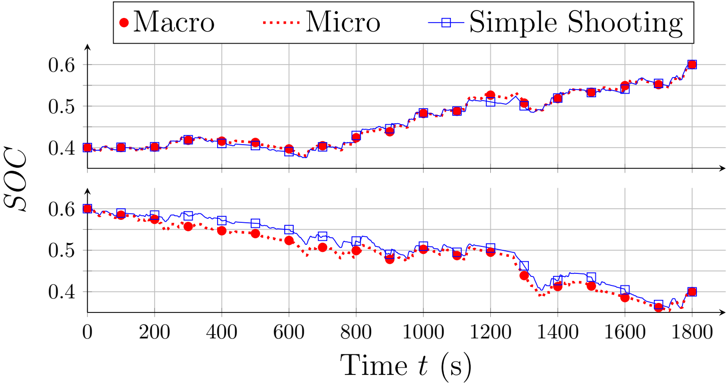

Figure: State trajectories of Macro-Micro and simple shooting methods.

Associated cost differences: 0.34g (0.039%) and 1.71g (0.244%).

Advantages for an embedded solution

The Macro-Micro method:

- is $N+1$ times faster than the simple shooting,
- needs $N+1$ times less computation than indirect methods,
- have small cost difference,
- is more robust with the natural initialization given by (3), see right side figure.

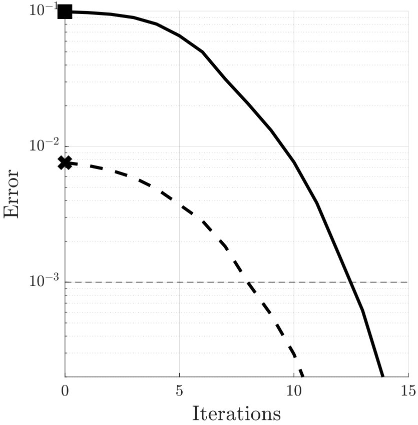

Figure: Evolution of the error $|S_1(\cdot)|$ w.r.t the number of iterations of a Newton-like solver (with 100 different initial and final states, on Δ_0).
—: fixed initialization $p = 500$ (■) - - -: natural initialization p^* (✖)
- - -: industrial tolerance 10^{-3}

Goal of the geometric preconditioner

Goal: further reducing the number of iterations of the solver.

Main idea[4]: preconditioning method of the shooting function based on

- a geometric interpretation of the costate,
- and the Mathieu transformation.

[4] cf. [Cots et al., 2024] for more information

Geometric interpretation of the costate

The proof of the maximum principle is constructive.

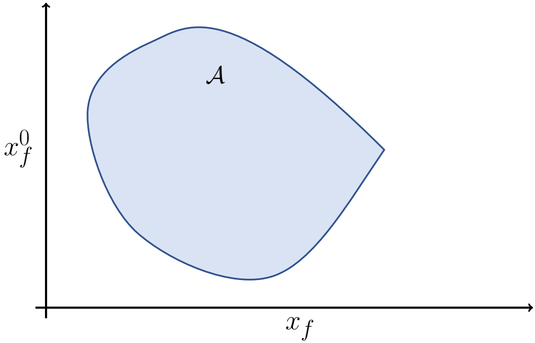

Figure: Illustration of the accessible augmented state set $\mathcal{A}$, which is the set of reachable augmented states $\hat{x}_f = (x_f^0, x_f)$ at t_f from x_0 at t_0.

Geometric interpretation of the costate

The proof of the maximum principle is constructive. The final augmented costate $\hat{p}_f = \left(p^0, p(t_f)\right)$ is taken in the polar of the proper convex Boltyanskii cone $\mathcal{K}^\circ$.

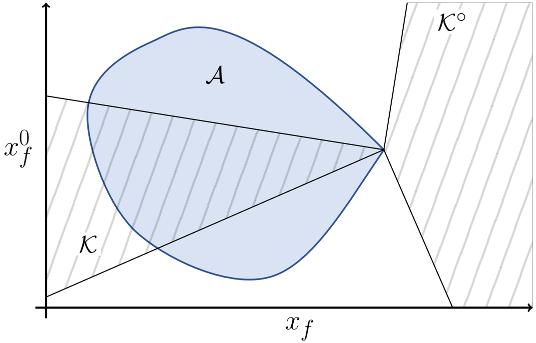

Figure: Illustration of the Botlyanskii cone $\mathcal{K}$ and its polar $\mathcal{K}^\circ$ at an augmented final state $\hat{x}_f \in \partial\mathcal{A}$.

Geometric interpretation of the costate

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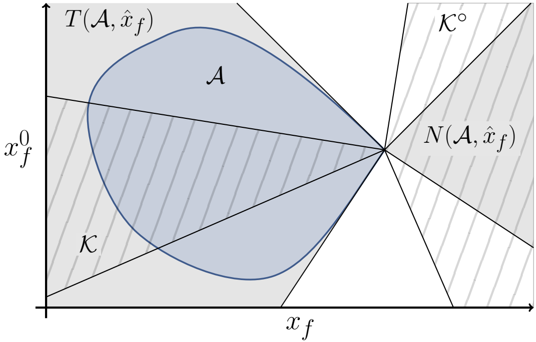
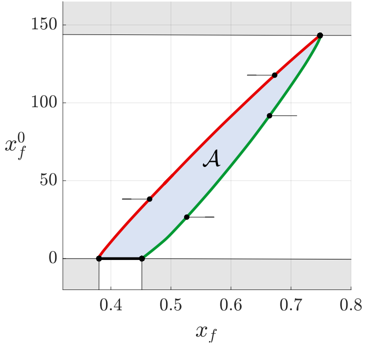
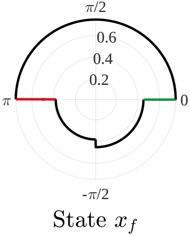


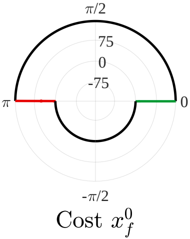
Figure: Illustration of the link between $\mathcal{K}^\circ$ and the normal cone $N\left(\mathcal{A}, \hat{x}_f\right)$ of the set $\mathcal{A}$ at the point $\hat{x}_f$.

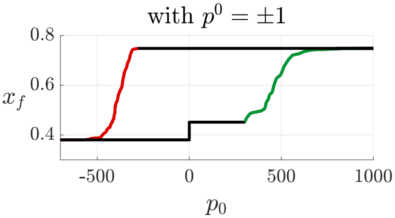
If $\mathcal{A}$ is closed and convex, we can take $\hat{p}(t_f) \in N\left(\mathcal{A}, \hat{x}_f\right)$.

Accessible augmented set and shooting functions









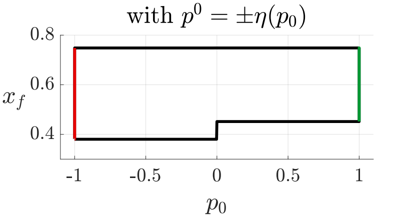


Figure: On Δ_0, with $x_0 = 0.5$.

Mathieu transformation

A diffeomorphism $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ on the augmented state is lifted into a diffeomorphism $\Phi\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ on the augmented state-costate that preserves the Hamiltonian dynamics

$$\Phi(\hat{x}, \hat{p}) = \big(\phi(\hat{x}), J_\phi(\hat{x})^{-\top} \hat{p}\big),$$

which is called Mathieu transformation.

This diffeomorphism transforms $\hat{z} = (\hat{x}, \hat{p})$ into $\hat{w} = (\hat{y}, \hat{q})$:

$$\hat{z} = \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} \begin{pmatrix} \hat{y} \\ \hat{q} \end{pmatrix} = \hat{w}.$$

Moreover, we denote $\hat{y} = (y^0, y)$ and $\hat{q} = (q^0, q)$.

Construction of the transformation

Main idea: fitting an ellipse on $\partial\mathcal{A}$ and creating the linear diffeomorphism $\phi(\hat{x}) = A\hat{x} + b$ that transforms this ellipse into the unit circle.

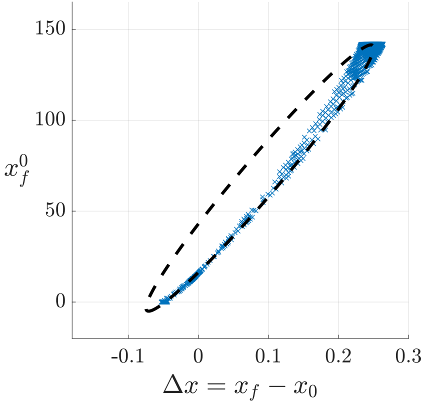


Figure: Original coordinates

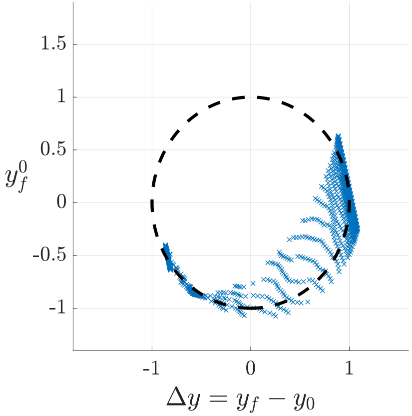


Figure: New coordinates

Construction of the transformation

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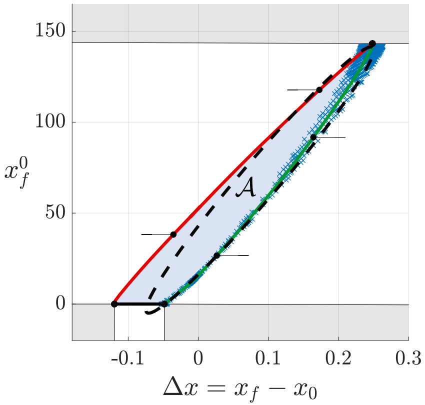

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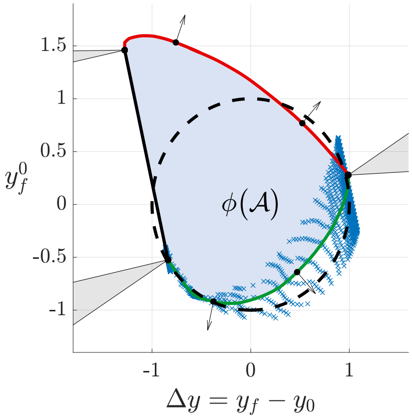
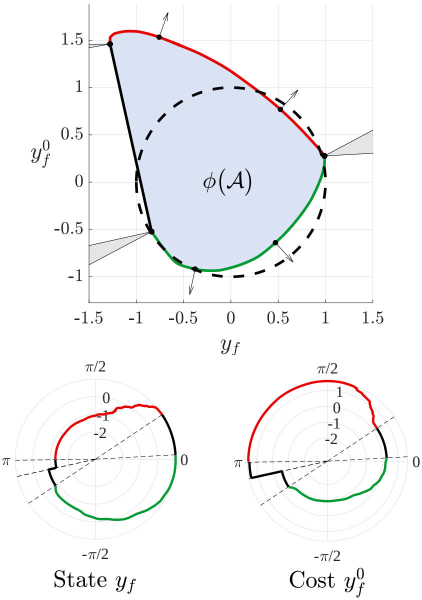


Figure: New coordinates

Geometric interpretation of the costate



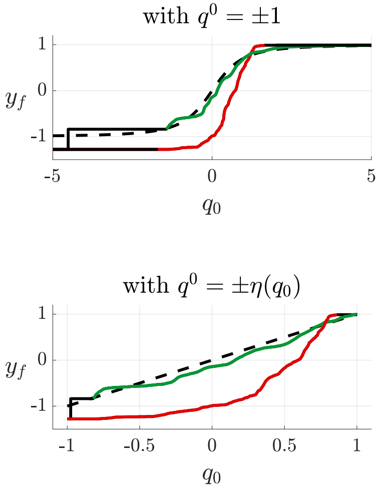


Figure: On Δ_0, with $x_0 = 0.5$.

Definition of the shooting functions

In the new coordinates, the shooting function $T \colon \mathbb{R}^- \times \mathbb{R} \to \mathbb{R}$ is given by

$$T(q^0, q) = \pi_y\big(\hat{y}_f(q^0, q)\big) - y_T$$

where π_y is the classical y-space projection, and the function $\hat{y}_f(\cdot)$ is constructed by

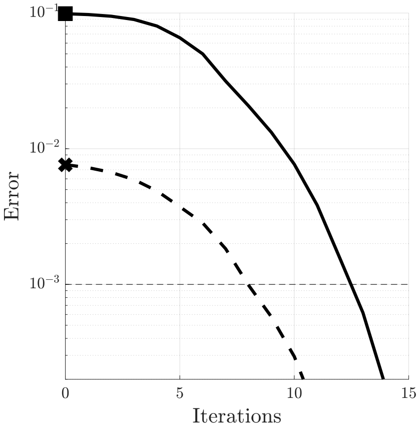
$$\begin{array}{ccc}
(p^0, p_0) = \hat{p}_0 & \xleftarrow[\hat{p}_0 = A^\top \hat{q}_0]{\hat{p}_0 = J_\phi(\hat{x}_0)^\top \hat{q}_0} & \hat{q}_0 = (q^0, q_0) \\
\Big\downarrow {\scriptstyle \hat{x}_f = \pi_{\hat{x}}\left(\exp_{\vec{\hat{h}}}(\cdot)\right)} & & \\
(x_f^0, x_f) = \hat{x}_f & \xrightarrow[A x_f + b = \hat{y}_f]{\phi(\hat{x}_f) = \hat{y}_f} & \hat{y}_f = (y_f^0, y_f)
\end{array}$$

The functions T_1 and T_2 are defined from T similarly as S_1 and S_2 from S.

Results

Figure: Evolution of the error w.r.t the number of iterations (with 100 different initial and final states, on Δ_0).

Init / Error	Fixed[5]	Natural
	■	✖
$\lvert S_1(\cdot)\rvert$	——	- - -

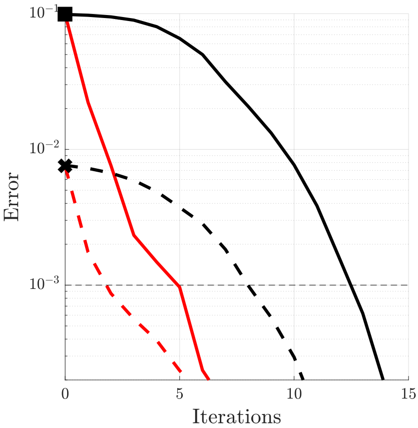


[5] $p = 500$ for S_1

Results

Figure: Evolution of the error w.r.t the number of iterations (with 100 different initial and final states, on Δ_0).

Error \ Init	Fixed[5] ■	Natural ✖
$\lvert S_1(\cdot) \rvert$	—— (black solid)	- - - (black dashed)
for T_2	—— (red solid)	- - - (red dashed)



The error for T_2 is converted into the original coordinates.

[5] $p = 500$ for S_1 and $q = 0$ for T_2.

Conclusion

Compared to classical indirect method, for an embedded solution, we proposed two methods.

- **Macro-Micro:**
  - $N + 1$ times faster,
  - needs $N + 1$ less computations,
  - small cost difference (<2g / <0.25%),
  - more robust with the natural initialization,

  compared to classical indirect method;

- **Geometric preconditioner:**
  - only 2 iterations of the solver in average,
  - no additional computational cost,
  - non-intrusive with respect to the model.

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Main property on the transformation

If $\phi \colon \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism then

$$\left\{ \begin{array}{l} \min\limits_{\hat{x}=(x^0,x)} \quad x^0, \\ \text{s.t. } \hat{x} \in \mathcal{A}, \\ \qquad x = x_T, \end{array} \right. \iff \left\{ \begin{array}{l} \min\limits_{\hat{y}=(y^0,y)} \quad \pi_{x^0}\big(\phi^{-1}(\hat{y})\big), \\ \text{s.t. } \hat{y} \in \phi\big(\mathcal{A}\big), \\ \qquad \pi_x\big(\phi^{-1}(\hat{y})\big) = x_T, \end{array} \right.$$

where π_{x^0} is the x^0-space projection. Moreover, if ϕ satisfy

$$\frac{\partial \phi}{\partial x^0} = \begin{pmatrix} k \\ 0 \end{pmatrix}, \quad k > 0, \tag{4}$$

then $\phi(\hat{x}) = \big(\phi_0(\hat{x}), \phi_x(x)\big)$ and

$$\left\{ \begin{array}{l} \min\limits_{\hat{x}=(x^0,x)} \quad x^0, \\ \text{s.t. } \hat{x} \in \mathcal{A}, \\ \qquad x = x_f, \end{array} \right. \iff \left\{ \begin{array}{l} \min\limits_{\hat{y}=(y^0,y)} \quad y^0, \\ \text{s.t. } \hat{y} \in \phi\big(\mathcal{A}\big), \\ \qquad y = y_T, \end{array} \right.$$

where $y_T = \phi_x(x_T)$.

Results

Figure: Evolution of the error

Black: $|S_1(\cdot)|$

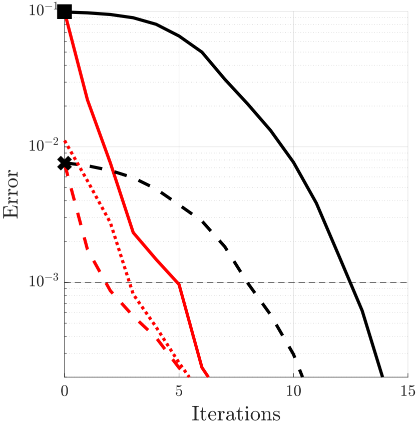
Red: for $T_2(\cdot)$

w.r.t. the number of iterations (with 100 different final states).

— / —: fixed init (■) ($p = 500$ / $q = 0$)

- - - / - - -: natural init (✖)

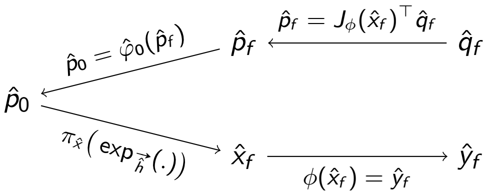
····· : $q = y_f$



The error for T_2 is calculated for each iterate on the initial coordinates.

Definition in the general case

In a general case, the function $\hat{y}_f(\cdot)$ is constructed by



where the function $\hat{\varphi}_0$ is an approximation of the map $\hat{p}_f \mapsto \hat{p}_0$. In our case, this approximation is the identity: