Geometric preconditioner for indirect method and application

Rémy Dutto [1,2,3]

[1]IRIT: Institut de Recherche en Informatique de Toulouse

[2]IMT: Institut de Mathématiques de Toulouse

[3]Vitesco Technologies

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Introduction

In collaboration with:

- Olivier Cots, IRIT, Toulouse,
- Olivier Flebus, Vitesco Technologies, Toulouse,
- Sophie Jan, IMT, Toulouse,
- Serge Laporte, IMT, Toulouse,
- Mariano Sans, Vitesco Technologies, Toulouse.

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We consider a Hybrid Electric Vehicle (HEV) on a predefined cycle, i.e. speed and slope trajectories are prescribed.

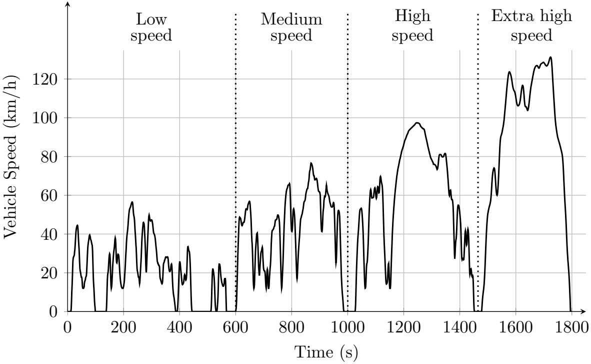


Figure: Worldwide harmonized Light vehicles Test Cycle (WLTC).

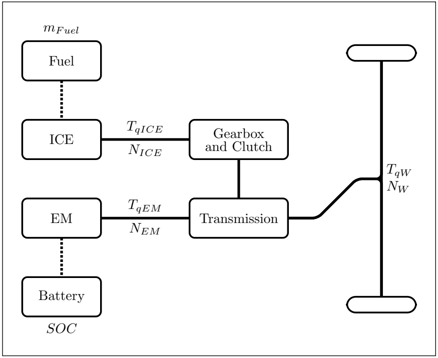
Requested wheels torque $T_{qW}(t)$ and rotation speed $N_W(t)$ are obtained with the information of our vehicle (mass, wheel diameter, aerodynamic coefficient...).

Static model of HEV

Inputs of our static model:

Name	Description	Unit
Cost		
m_{Fuel}	Fuel consumption	g
State		
SOC	Battery state of charge	
Commands		
$Gear$ T_{qICE}	Gearbox selector ICE torque	N.m
External inputs		
T_{qW} N_W	Wheels torque Wheels rotation speed	N.m RPM

Figure: Schema of the selected HEV.



Outputs: $\dot{m}_{Fuel}$ and $\dot{SOC}$, where $\dot{}$ stands for $\dfrac{\mathrm{d}}{\mathrm{d}t}$.

HEV torque split and gear shift problem

The Hybrid Electric Vehicle torque split and gear shift problem can be formulated as a classical Lagrange optimal control problem

$$
\text{(OCP)} \quad \begin{cases} V(x_0, x_f) = \min\limits_{x,u} \displaystyle\int_{t_0}^{t_f} f^0\big(t, x(t), u(t)\big)\,\mathrm{d}t & \\ \text{s.t. } \dot{x}(t) = f\big(t, x(t), u(t)\big) & t \in [t_0, t_f] \text{ a.e.}, \\ \qquad u(t) \in \mathrm{U}(t) & t \in [t_0, t_f], \\ \qquad x(t_0) = x_0, \quad x(t_f) = x_f, & \end{cases}
$$

where:

- $x \in \mathrm{AC}([t_0, t_f], \mathbb{R})$ corresponds to the *SOC*,
- $u \in \mathrm{L}^\infty([t_0, t_f], \mathbb{R}^2)$ corresponds to the pair $\big(T_{qICE}, Gear\big)$,
- functions f^0 and f are $\mathcal{C}^1$ w.r.t. x and u,
- $\mathrm{U}(t) \subset \mathbb{R}^2$ is a nonempty closed set for every $t \in [t_0, t_f]$, with regularity assumptions.[1]

[1](cf. [Cesari, 1983, Chapter 4.2, Remark 5] for more information)

Augmented system

Motivated by [Cots et al., 2023], we only consider the first 100s of the cycle ($[t_0, t_f] = [0, 100]$).

We propose to consider the augmented formulation of (OCP)

$$
\text{(AOCP)} \qquad \begin{cases} \displaystyle\min_{\hat{x}, u} x^0(t_f) \\ \text{s.t. } \dot{\hat{x}}(t) = \hat{f}\big(t, \hat{x}(t), u(t)\big) & t \in [t_0, t_f] \text{ a.e.}, \\ \qquad u(t) \in \mathrm{U}(t) & t \in [t_0, t_f], \\ \qquad \hat{x}(t_0) = \hat{x}_0, \quad x(t_f) = x_f, \end{cases}
$$

where $\hat{f} \colon \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$ is the augmented system

$$
\hat{f}(t, \hat{x}, u) = \big(f^0(t, x, u), f(t, x, u)\big)
$$

and where $\hat{x} = (x^0, x)$ corresponds to the cost-state pair, with $\hat{x}_0 = (0, x_0)$.

Pontryagin's Maximum Principle

If $(\hat{x}, u)$ is solution of (AOCP), there exists an augmented costate $\hat{p} = (p^0, p) \in \mathrm{AC}([t_0, t_f], \mathbb{R}^2)$ such that

$$p^0 \leq 0, \quad \hat{p} \neq 0, \tag{1}$$

the Hamilton's dynamic is satisfied

$$\begin{aligned} \dot{\hat{x}}(t) &= \quad \nabla_{\hat{p}} h\big(t, \hat{x}(t), \hat{p}(t), u(t)\big) \quad t \in [t_0, t_f] \text{ a.e.}, \\ \dot{\hat{p}}(t) &= -\nabla_{\hat{x}} h\big(t, \hat{x}(t), \hat{p}(t), u(t)\big) \quad t \in [t_0, t_f] \text{ a.e.}, \end{aligned}$$

and the maximization condition is satisfied

$$h\big(t, \hat{x}(t), \hat{p}(t), u(t)\big) = \max_{w \in \mathrm{U}(t)} h\big(t, \hat{x}(t), \hat{p}(t), w\big) \quad t \in [t_0, t_f] \text{ a.e.},$$

where $h(t, \hat{x}, \hat{p}, u) = \big(\hat{p} \,\big|\, \hat{f}(t, \hat{x}, u)\big)$ is the pseudo-Hamiltonian of the augmented system.

Notations

For the following presentation, we denote

$$\hat{x} = (x^0, x) \quad \text{and} \quad \hat{p} = (p^0, p).$$

Moreover, we denote

$$\hat{z} = (\hat{x}, \hat{p}) \quad \text{and} \quad z = (x, p).$$

These notations can be used for absolutely continuous functions or for vectors.

Since $\hat{f}$ does not depend on the cost x^0, we obtain $\dot{p}^0(\cdot) = 0$ and thus $p^0(\cdot) = p^0(t_0)$.

Hamiltonian framework

An extremal is a function $\hat{z} \in \mathrm{AC}([\mathrm{t_0}, \mathrm{t_f}], \mathbb{R}^4)$ that satisfies Equation (1), the Hamilton's dynamic and the maximization condition.

We consider that the Hamiltonian

$$H(t, \hat{z}) = \max_{u \in \mathrm{U}(t)} h(t, \hat{z}, u)$$

is $\mathcal{C}^1$ in a neighborhood of a given reference extremal. Under this assumption, the Hamiltonian vector field is defined by

$$\vec{H}(t, \hat{z}) = \big(\nabla_{\hat{p}} H(t, \hat{z}), -\nabla_{\hat{x}} H(t, \hat{z})\big),$$

and we get the following proposition

Proposition 1 ([Agrachev and Sachkov, 2004], Proposition 12.1)

$\hat{z}$ *is an extremal of* (AOCP) *if and only if Equation* (1) *is satisfied and*

$$\dot{\hat{z}}(t) = \vec{H}\big(t, \hat{z}(t)\big).$$

Simple shooting method

Under the previous assumption, the maximum principle leads to the resolution of

$$x_f = \pi_x\big(\exp_{\vec{H}}(\hat{x}_0, \hat{p}_0)\big), \quad p^0 \leq 0,$$

where $\pi_x(\cdot)$ is the classical x-space projection and the exponential map $\exp_{\vec{H}}(\hat{z}_0)$ of a field $\vec{H}$ is the solution at time t_f of the Cauchy problem

$$\dot{\hat{z}}(t) = \vec{H}\big(t, \hat{z}(t)\big), \quad t \in [t_0, t_f], \quad \hat{z}(t_0) = \hat{z}_0.$$

The simple shooting methods aim to find a non-trivial initial costate $\hat{p}$ where the shooting function

$$\begin{array}{rcccc} S & : & \mathbb{R}^- \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ & & \hat{p} & \longmapsto & \pi_x\big(\exp_{\vec{H}}(\hat{x}_0, \hat{p})\big) \end{array}$$

is equal to x_f.

Normalization of the shooting function

Let's remark that if $\hat{p} = (p^0, p)$ satisfies $S(\hat{p}) = x_f$ then for all $k > 0$, $S(k\hat{p}) = x_f$ (due to homogeneity of extremals $(\hat{x}, \hat{p})$ on $\hat{p}$).

We propose three shooting functions.
If we assume that the extremals associated to a solution are normal ($p^0 < 0$), then we can fix $p^0 = -1$ and consider $S_1 \colon \mathbb{R} \to \mathbb{R}$ defined by

$$S_1(p) = S(-1, p).$$

Without further assumption, we can fix $\|\hat{p}\|_2 = 1$ and consider $S_\alpha \colon [-\pi, 0] \to \mathbb{R}$ defined by

$$S_\alpha(\alpha) = S(\sin\alpha, \cos\alpha)$$

or $S_2 \colon [-1, 1] \to \mathbb{R}$ defined by

$$S_2(p) = S\big(\eta(p), p\big), \quad \text{where} \quad \eta(p) = -\sqrt{1 - p^2}.$$

Shooting functions

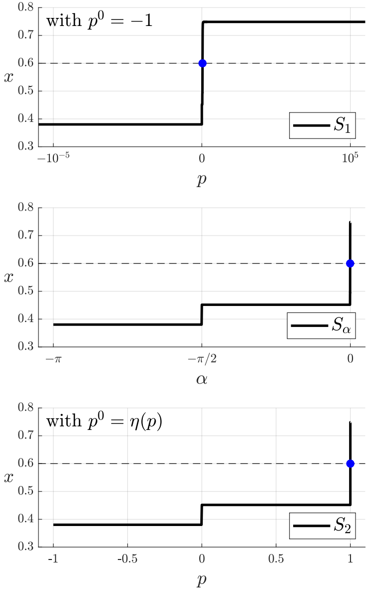


Figure: Evolution of the final state x with respect to

- the initial costate p with $p^0 = -1$ (S_1),
- the initial angle α (S_α),
- the initial costate p with $p^0 = \eta(p)$ (S_2).

The dashed line corresponds to $x = x_f$, and the blue point to the solution.

Shooting functions

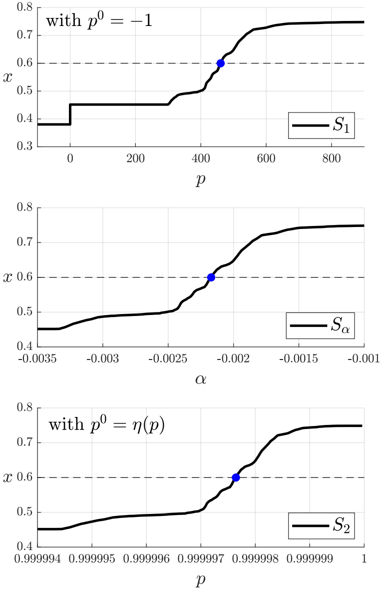


Figure: Evolution of the final state x with respect to

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- the initial costate p with $p^0 = \eta(p)$ (S_2).

The dashed line corresponds to $x = x_f$, and the blue point to the solution.

Results

A solution of the shooting method is found by a Newton-like solver.

Motivated by [Cots et al., 2023], we assume that we know an approximation C of the value function V.

A natural initial guess for S_1 is given by

$$p_* = -\nabla_{x_0} C(x_0, x_f).$$

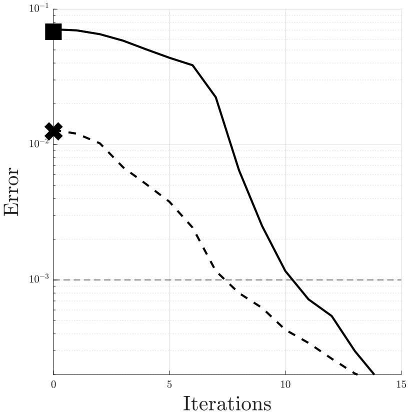

Figure: Evolution of the error $|S_1(\cdot) - x_f|$ w.r.t the number of iterations (with 100 different final states).

——: fixed initialization $p = 500$ (■) - - -: natural initialization p_* (✖)

- - -: industrial tolerance 10^{-3}

Goal

Goal: Reducing the number of iterations of the solver

Main idea[2]: Preconditioning method of the shooting function based on

- a geometric interpretation of the costate
- and the Mathieu transformation.

[2] cf. [Cots et al., 2024] for more information

Geometric interpretation of the costate

The proof of the maximum principle is constructive.

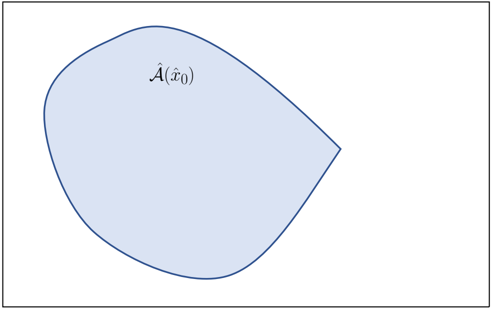

Figure: Illustration of the accessible augmented state $\hat{\mathcal{A}}(\hat{x}_0)$, which corresponds to the set of reachable augmented states $\hat{x}_f$ at time t_f from $\hat{x}_0$.

Geometric interpretation of the costate

The proof of the maximum principle is constructive. The final costate $\hat{p}(t_f)$ is taken in the polar of the proper convex Boltyanskii cone $\hat{\mathcal{K}}^\circ(\hat{x}_0, \hat{x}_f)$.

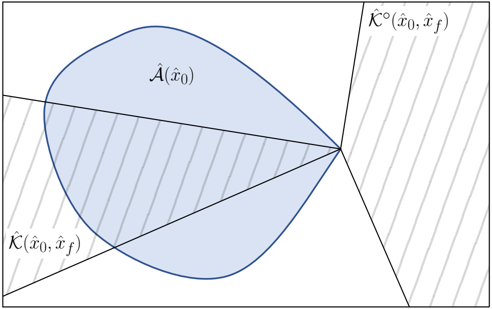

Figure: Illustration of the proper convex Botlyanskii cone $\hat{\mathcal{K}}(\hat{x}_0, \hat{x}_f)$ and its polar $\hat{\mathcal{K}}^\circ(\hat{x}_0, \hat{x}_f)$ at a point $x_f \in \partial\hat{\mathcal{A}}(\hat{x}_0)$.

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Geometric interpretation of the costate

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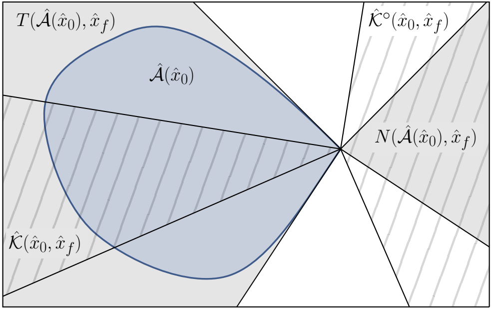
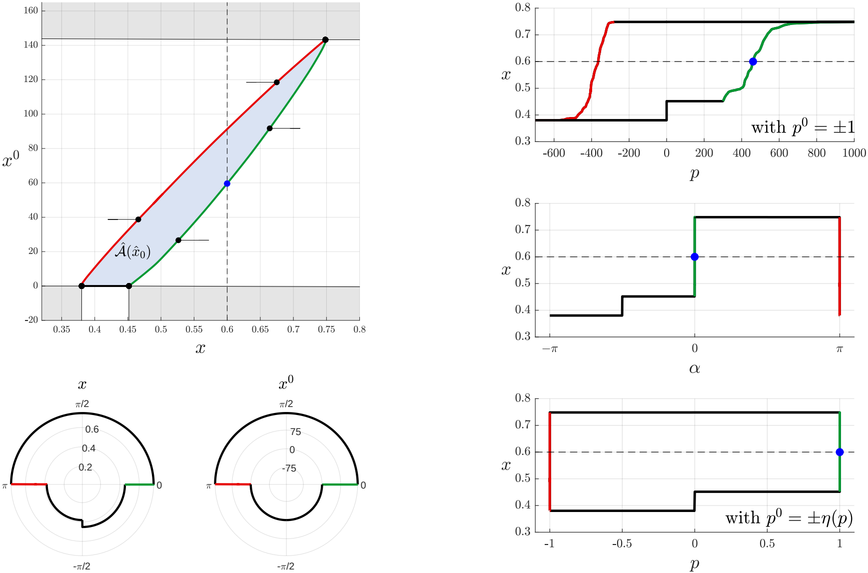


Figure: Illustration of the link between $\hat{\mathcal{K}}^\circ(\hat{x}_0, \hat{x}_f)$ and the normal cone $N(\hat{\mathcal{A}}(\hat{x}_0), \hat{x}_f)$ of the admissible augmented state $\hat{\mathcal{A}}(\hat{x}_0)$ at the point $\hat{x}_f$.

If $\hat{\mathcal{A}}(\hat{x}_0)$ is closed and convex, we can take $\hat{p}(t_f) \in N(\hat{\mathcal{A}}(\hat{x}_0), \hat{x}_f)$.

Geometric interpretation of the costate



Mathieu transformation

A diffeomorphism $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ on the augmented state is lifted into a diffeomorphism $\Phi\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ on the augmented state-costate that preserves the Hamiltonian dynamics by the so-called Mathieu transformation

$$\Phi(\hat{x}, \hat{p}) = \big(\phi(\hat{x}), J_\phi(\hat{x})^{-\top} \hat{p}\big).$$

This diffeomorphism transforms $\hat{z} = (\hat{x}, \hat{p})$ into $\hat{w} = (\hat{y}, \hat{q})$:

$$\hat{z} = \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} \begin{pmatrix} \hat{y} \\ \hat{q} \end{pmatrix} = \hat{w}.$$

Moreover, we denote $\hat{y} = (y^0, y)$ and $\hat{q} = (q^0, q)$.

Construction of the transformation

Idea: Fitting an ellipse on $\partial\hat{\mathcal{A}}(\hat{x}_0)$ and creating the linear diffeomorphism ϕ that transforms this ellipse into the unit circle.

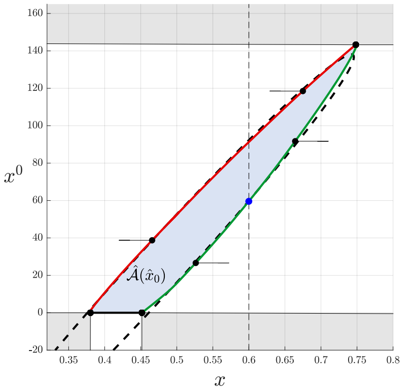


Figure: Initial coordinates

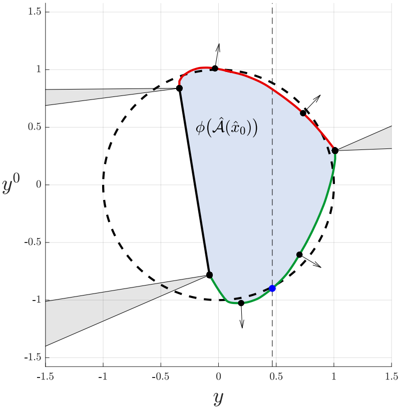
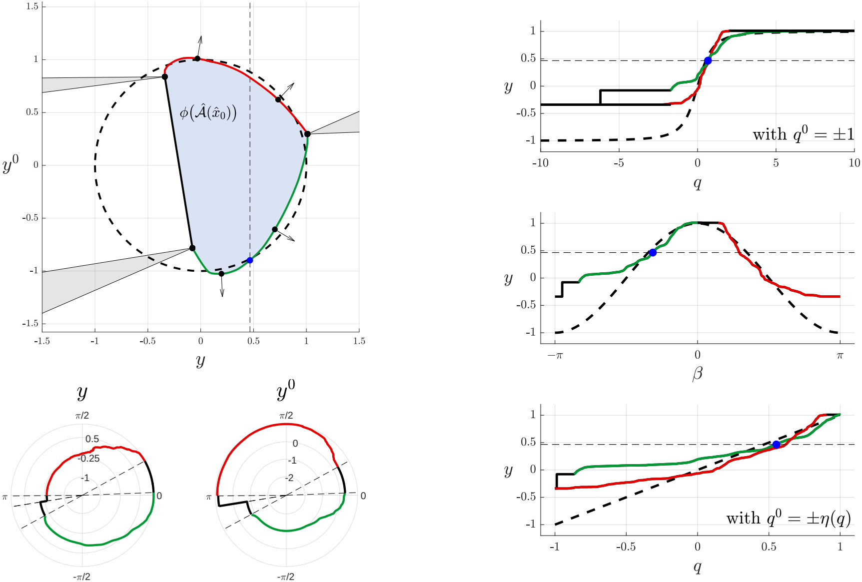


Figure: Final coordinates

Geometric interpretation of the costate



Definition of the shooting functions

In the new coordinates, the shooting function $T\colon \mathbb{R}^- \times \mathbb{R} \to \mathbb{R}$ is defined by

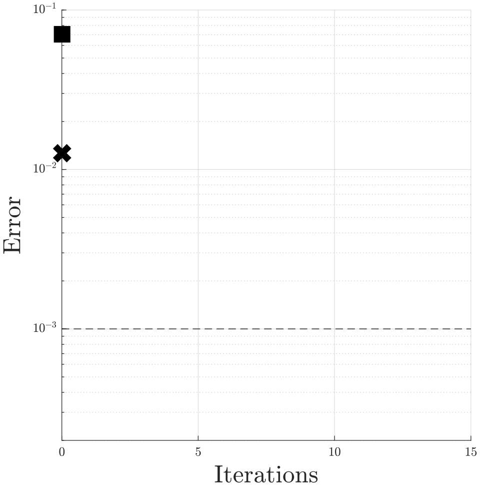
$$T(\hat{q}) = \pi_y\Big(\overbrace{\Phi \circ \exp_{\vec{H}}\big(\hat{x}_0, \underbrace{J_\phi(\hat{x}_0)^\top \hat{q}}_{\hat{p}}\big)}^{\hat{w}=(\hat{y},\hat{q})}\Big)$$

where π_y is the classical y-space projection. The functions T_1, T_β and T_2 are defined with T by the same way than S_1, S_α and S_2 with S.

Results

Figure: Evolution of the error w.r.t the number of iteration (with 100 different final states).

Init / Error	Fixed[3] ■	Natural ✖
$\lvert S_1(\cdot) - x_f \rvert$		
$\lvert \phi_x^{-1}(T_2(\cdot)) - x_f \rvert$		

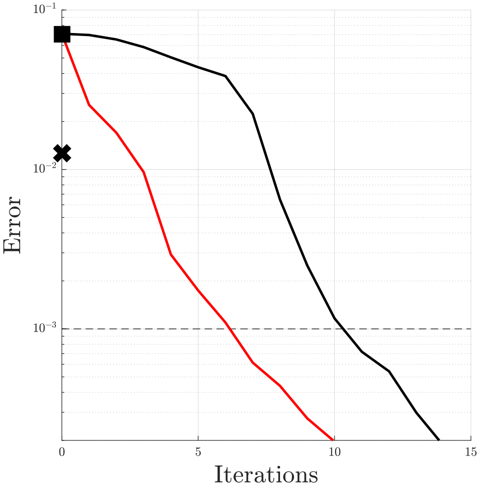


[3] $p = 500$ for S_1 and $q = 0$ for T_2.

Results

Figure: Evolution of the error w.r.t the number of iteration (with 100 different final states).

Init / Error	Fixed[3]	Natural
	■	✖
$\left\|S_1(\cdot) - x_f\right\|$	— (black)	
$\left\|\phi_x^{-1}\left(T_2(\cdot)\right) - x_f\right\|$	— (red)	

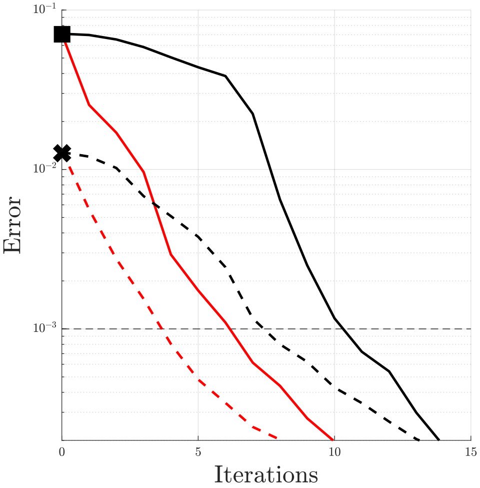


[3] $p = 500$ for S_1 and $q = 0$ for T_2.

Results

Figure: Evolution of the error w.r.t the number of iteration (with 100 different final states).

Error \ Init	Fixed[3] ■	Natural ✖
$\lvert S_1(\cdot) - x_f \rvert$	solid black line	dashed black line
$\lvert \phi_x^{-1}(T_2(\cdot)) - x_f \rvert$	solid red line	dashed red line

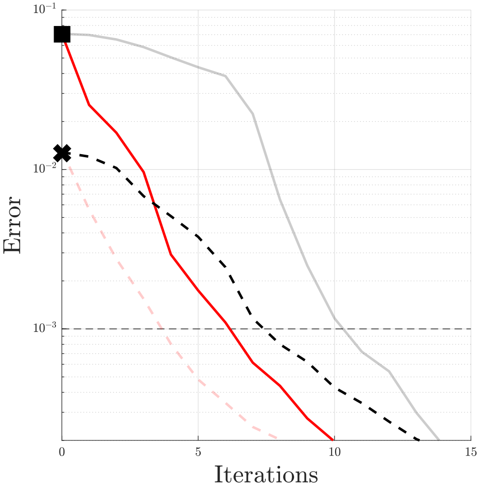


[3] $p = 500$ for S_1 and $q = 0$ for T_2.

Results

Figure: Evolution of the error w.r.t the number of iteration
(with 100 different final states).

Error \ Init	Fixed[3] ■	Natural ✖
$\left\lvert S_1(\cdot) - x_f \right\rvert$	grey solid line	black dashed line
$\left\lvert \phi_x^{-1}\left(T_2(\cdot)\right) - x_f \right\rvert$	red solid line	pink dashed line



[3] $p = 500$ for S_1 and $q = 0$ for T_2.

Conclusion

We proposed a new preconditioning method of the shooting function:

- based on a geometric interpretation of the costate and on the Mathieu transformation,
- which reduces the number of iterations of the solver,
- which is not intrusive with the model,
- that needs an estimation of the accessible augmented state $\hat{\mathcal{A}}(\hat{x}_0)$.

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Main property on the transformation

If $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism then

$$\left\{\begin{array}{l} \min\limits_{\hat{x}} x^0, \\ \text{s.t. } \hat{x} \in \hat{\mathcal{A}}(\hat{x}_0), \\ \qquad x = x_f, \end{array}\right. \iff \left\{\begin{array}{l} \min\limits_{\hat{y}} \pi_{x^0}\big(\phi^{-1}(\hat{y})\big), \\ \text{s.t. } \hat{y} \in \phi\big(\hat{\mathcal{A}}(\hat{x}_0)\big), \\ \qquad \pi_x\big(\phi^{-1}(\hat{y})\big) = x_f, \end{array}\right.$$

where π_{x^0} is the x^0-space projection. Moreover, if ϕ satisfy

$$\frac{\partial \phi}{\partial x^0} = \begin{pmatrix} k \\ 0 \end{pmatrix}, \quad k > 0, \tag{2}$$

then $\phi(\hat{x}) = \big(\phi_0(\hat{x}), \phi_x(x)\big)$ and

$$\left\{\begin{array}{l} \min\limits_{\hat{x}} x^0, \\ \text{s.t. } \hat{x} \in \hat{\mathcal{A}}(\hat{x}_0), \\ \qquad x = x_f, \end{array}\right. \iff \left\{\begin{array}{l} \min\limits_{\hat{y}} y^0, \\ \text{s.t. } \hat{y} \in \phi\big(\hat{\mathcal{A}}(\hat{x}_0)\big), \\ \qquad y = y_f, \end{array}\right.$$

where $y_f = \phi_x(x_f)$.

Results

Figure: Evolution of the error

Black : $|S_1(\cdot) - x_f|$

Red : $|\phi_x^{-1}(T_2(\cdot)) - x_f|$

w.r.t. the number of iterations (with 100 different final states).

— / —: fixed init (■) ($p = 500$ / $q = 0$)

- - - / - - -: natural init (✖)

⋯⋯ : $q = y_f$