Indirect method in optimal control

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Introduction

Direct method : « First discretize then optimize. »

Indirect method : « First optimize then discretize. »

The main idea is to first apply the Pontryagin maximum principle and then numerically solve the resulting problem.

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Optimal control problem

We consider the following optimal control problem

$$
\text{(OCP)} \qquad \left\{
\begin{array}{ll}
\displaystyle \min_{x,u} \int_{t_0}^{t_f} f^0\big(t, x(t), u(t)\big)\,\mathrm{d}t, & \\
\text{s.c. } \dot{x}(t) = f\big(t, x(t), u(t)\big), & t \in [t_0, t_f] \text{ a.e.}, \\
\qquad u(t) \in \mathrm{U}(t), & t \in [t_0, t_f], \\
\qquad c(x(t_0), x(t_f)) = 0, &
\end{array}
\right.
$$

where $f^0 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is the objective function, $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the dynamics, $\mathrm{U}(t) \subset \mathbb{R}^m$ is the control set, and $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ is the boundary condition.

Functions f^0, f and c are assumed to be smooth (at least $\mathcal{C}^1$) and c is a submersion on $M = c^{-1}(\{0\})$.

Pontryagin Maximum Principle

Theorem 1

If (x, u) is a solution of (OCP) then there exists a non-trivial pair $(p_0, p) \in \mathbb{R}^- \times \mathrm{AC}([t_0, t_f], \mathbb{R}^n)$ such that the Hamiltoninan dynamic is satisfied for a.e. $t \in [t_0, t_f]$:

$$\dot{x}(t) = \nabla_p h\big(t, x(t), p(t), p^0, u(t)\big),$$

$$\dot{p}(t) = -\nabla_x h\big(t, x(t), p(t), p^0, u(t)\big),$$

as well as the maximization condition for a.e. $t \in [t_0, t_f]$:

$$h\left(t, x(t), p(t), p^0, u(t)\right) = \max_{w \in \mathrm{U}(t)} h\left(t, x(t), p(t), p^0, w\right),$$

and the transversality condition :

$$\big(-p(t_0), p(t_f)\big) \perp T_{(x(t_0), x(t_f))} M,$$

where h is the pseudo-Hamiltonian defined by

$$h(t, x, p, p^0, u) := p^0 f^0(t, x, u) + \big(p \mid f(t, x, u)\big).$$

Hamiltonian dynamic

Denoting $z = (x, p) \in \mathrm{AC}([t_0, t_f], \mathbb{R}^{2n})$ the state-costate pair, the Hamiltonian dynamic is defined by

$$\dot{z}(t) = \overrightarrow{h}(t, z(t), p^0, u(t)),$$

where $\overrightarrow{h}$ is the Hamiltonian vector field defined by

$$\overrightarrow{h}(t, z, p^0, u) = J \nabla_z h(t, z, p^0, u),$$

and where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ is the symplectic matrix.

Transversality conditions

For all $v \in M := c^{-1}(\{0\})$, one has $T_v M = \mathrm{Ker}(c'(v))$.

Denoting $(b_1(v), \ldots, b_{2n-p}(v))$ a basis of $\mathrm{Ker}(c'(v))$ and $B(v) = \begin{bmatrix} b_1(v) & \ldots & b_{2n-p}(v) \end{bmatrix}$ the associated matrix, we have

$$u \perp T_v M \iff B(v)^\top u = 0.$$

This means that the transversality conditions can be written as $c^\star(z(t_0), z(t_f)) = 0$, where $c^\star \colon \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n-p}$ is defined by

$$c^\star(z_0, z_f) = B(x_0, x_f)^\top \begin{bmatrix} -p_0 \\ p_f \end{bmatrix}.$$

We can regroup the state and costate boundary conditions on the function $g \colon \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined by

$$g(z_0, z_f) = \big(c(x_0, x_f), c^\star(z_0, z_f)\big).$$

Exponential map

Hypothesis 1

The function $\exp_{\vec{h}}(t_2, t_1, z_0, p^0) \mapsto z(t_2)$ *where* $z(\cdot)$ *is solution of*

$$\begin{cases} \dot{z}(t) = \vec{h}\big(t, z(t), p^0, u(t)\big), & t \in [t_1, t_2] \text{ a.e.}, \\ h\big(t, z(t), p^0, u(t)\big) = \max_{w \in \mathrm{U}(t)} h\big(t, z(t), p^0, w\big), & t \in [t_1, t_2] \text{ a.e.} \\ z(t_1) = z_0. & \end{cases}$$

is an application, defined for all $t_0 \leq t_1 < t_2 \leq t_f$, *for all* $z_0 \in \mathbb{R}^{2n}$ *and for all* $p^0 \leq 0$.

Such hypothesis is verified for the strict Legendre framework ($\frac{\partial^2 h}{\partial u^2} \prec 0$), for the Hamiltonian framework (maximized pseudo-Hamiltonian is a smooth true Hamiltonian), and even more.

Main idea of the simple shooting method

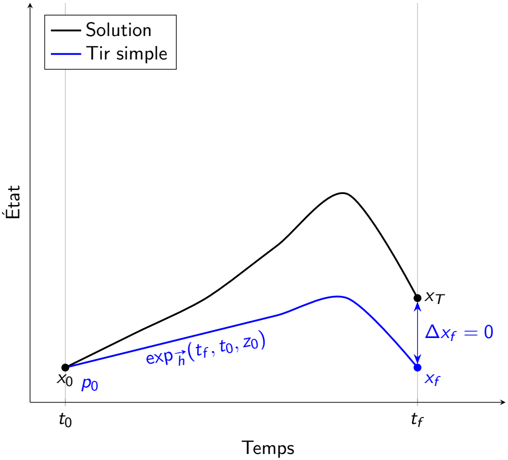

Figure: Simple shooting method illustration.

Simple shooting method

Under Hypothesis 1, the Pontryagin maximum principle leads to the resolution of the following Two Points Boundary Value Problem

$$(\text{TPBVP}) \quad \begin{cases} z_f = \exp_{\vec{h}}(t_f, t_0, z_0, p^0), \\ g(z_0, z_f) = 0, \quad p^0 \leq 0, \quad \left(p^0, \pi_p(z_0)\right) \neq 0. \end{cases}$$

where $\pi_p(x, p) = p$ is the projection on the co-state variable.

The simple shooting method consists in finding a non-trivial zero of the simple shooting function $S\colon \mathbb{R}^{2n} \times \mathbb{R}^- \to \mathbb{R}^{2n}$ defined by

$$S(z_0, p^0) = g(z_0, \exp_{\vec{h}}(t_f, t_0, z_0, p^0)).$$

Normalisation of the shooting function

Remark (Homogeneity)

$$S(x_0, p_0, p^0) = 0 \quad \Rightarrow \quad \forall k > 0, \quad S(x_0, kp_0, kp^0) = 0.$$

We propose two methods of normalization of S.

- Method 1 : if we suppose that the extremals associated to the solutions of (OCP) are normal ($p^0 < 0$), then we can fix $p^0 = -1$ and consider $S_1 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined by

$$S_1(z_0) = S(z_0, -1),$$

- Method 2 : without assuming the normality of the extremals, we can fix $\|(p^0, p_0)\|_2 = 1$ and consider $S_2 : \mathbb{R}^n \times \mathcal{B}^n \to \mathbb{R}^{2n}$ defined by

$$S_2(x_0, p_0) = S\big(x_0, p_0, \eta(p_0)\big), \quad \text{where} \quad \eta(p_0) = -\sqrt{1 - \|p_0\|_2^2}.$$

Problem with structure : Numerical examples

A Numerical example of optimal control problem with structure :

Turnpike example:
https://remydutto.github.io/TurnpikeExample.jl/dev/direct-indirect.html

Membrane filtration system:
https://remydutto.github.io/CTMembraneFiltration.jl/stable/index.html

Advantages and drawbacks

Compared to direct method, indirect simple shooting mainly has the following advantages and drawbacks:

- Advantages :
  - Low-dimensional optimization problem.
  - High precision of the solution.
- Drawbacks :
  - Need the knowledge of the structure of the problem.
  - Simple shooting function is known to be sensitive to the initial guess.

Questions : Can we have a « good » initial guess ? How to reduce the sensitivity of the shooting function ?

Initial guess

The most classical way to obtain an initial guess is to use a direct method : it provides both structure and a numerical solution.

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If we know an approximation $C(a,b)$ of the value function $V(a,b)$ defined by

$$\begin{cases} V(a,b) := \min_{x,u} \int_{t_0}^{t_f} f^0\big(t, x(t), u(t)\big)\,\mathrm{d}t, & \\ \text{s.t. } \dot{x}(t) = f\big(t, x(t), u(t)\big), & t \in [t_0, t_f] \text{ a.e.}, \\ \quad u(t) \in \mathrm{U}(t), & t \in [t_0, t_f], \\ \quad x(t_0) = a, \quad x(t_f) = b, & \end{cases}$$

one can use $\nabla C(a,b)$ to construct a good initial guess since

$$\nabla_a V\big(x(t_0), x(t_f)\big) = -p(t_0) \quad \text{and} \quad \nabla_b V\big(x(t_0), x(t_f)\big) = p(t_f).$$

Main idea of the multiple shooting method

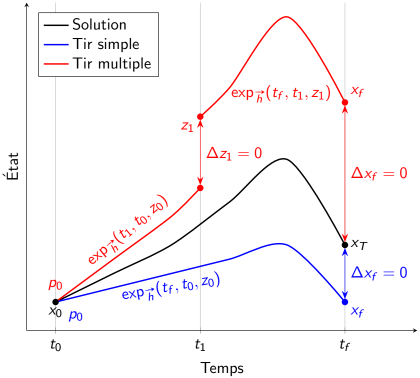


Figure: Simple and multiple shooting method illustration.

Multiple shooting method

The time interval $[t_0, t_f]$ is decomposed into $\Delta_i = [t_i, t_{i+1}]$, where $t_0 < t_1 < \cdots < t_N < t_{N+1} = t_f$.

Problem (TPBVP) is transformed into

$$(\text{MPBVP}) \qquad \begin{cases} \forall i \in \mathbb{N}_{N-1}, \quad z_{i+1} = \exp_{\vec{h}}(t_{i+1}, t_i, z_i, p^0), \quad p^0 \leq 0, \\ g(z_0, \exp_{\vec{h}}(t_{N+1}, t_N, z_N, p^0)) = 0. \end{cases}$$

The multiple shooting function is defined by

$$(z_0, \ldots, z_N, p^0) \longmapsto \begin{pmatrix} \exp_{\vec{h}}\left(t_1, t_0, z_0, p^0\right) - z_1 \\ \vdots \\ \exp_{\vec{h}}\left(t_N, t_{N-1}, z_{N-1}, p^0\right) - z_N \\ g(z_0, \exp_{\vec{h}}(t_{N+1}, t_N, z_N, p^0)) \end{pmatrix}.$$

This function is known to be less sensitive to the initial guess than the function S.

Geometric preconditioner

For now, let suppose that $c(a, b) = (a - x_0, b - x_f)$, with $x_0, x_f \in \mathbb{R}^n$ and that the augmented accessible set $\mathcal{A}$ is closed and convex. The set $\mathcal{A}$ is defined as the set of augmented state $\hat{x} = (x^0, x)$ reachable at the final time t_f, where x^0 is the cost trajectory.

In this case, the simple shooting function is defined by

$$S(p_0, p^0) = \pi_x(\exp_{\vec{h}}(t_f, t_0, x_0, p_0, p^0)) - x_f.$$

where $\pi_x(x, p) = x$ is the projection on the state variable.

Geometric preconditioner

Main idea: Fit an ellipse on the boundary $\partial\mathcal{A}$ of the augmented accessible set $\mathcal{A}$ and construct the linear diffeomorphism $\phi(\hat{x}) = A\hat{x} + b$ which transforms this ellipse into the unit circle, and which satisfy

$$\frac{\partial \phi}{\partial x^0} = (k, 0), \quad \text{with} \quad k > 0, \quad \text{i.e.} \quad A = \begin{bmatrix} k & a_{x^0} \\ 0 & a_x \end{bmatrix}$$

where $\hat{x} = (x^0, x)$ is the augmented state, with $a_{x^0} \in \mathcal{M}_{1,n}(\mathbb{R})$ and $a_x \in \mathcal{M}_{n,n}(\mathbb{R})$.
One can defined the shooting function

$$T(\hat{q}_0) = a_x S(A^\top \hat{q}_0),$$

and T_1 and T_2 with T in the same way than S_1 and S_2 with S.

Geometric preconditioner

Proposition 1

If $\phi(\mathcal{A})$ is the unit ball[1] then the shooting function T_2 is defined by

$$T_2(q_0) = q_0 - y_T,$$

where $y_T = a_x x_T + b_x$.

Toy example:
https://remydutto.github.io/GeometricPreconditioner.jl/dev/2D-preconditioner.html

[1] and under another assumption

Industrial application

Torque split and gear shift of Hybrid Electric Vehicle (HEV):

Figure: Evolution of the error

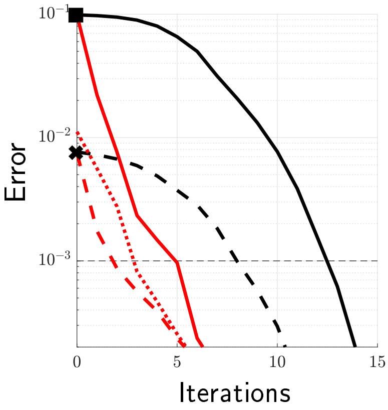
black : $|S_1(\cdot)|$

red : for $T_2(\cdot)$

with respect to the number of iterations (100 different initial and final states).

Initialization :

- —— / ——: fixed (■) ($p = 500$ / $q = 0$)
- - - - / - - -: natural (✖)
- ····· : $q = y_f$



Thank you for your attention !

https://remydutto.github.io/